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Approximate Reanalysis Based on the Exact Analytic Expressions

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The first author has recently given the exact analytic expressions of the inverse of the stiffness matrix, the nodal displacements, and the stress resultants in linear elastic structures composed of prismatic elements. For structures of constant geometry, the expressions are explicit in terms of the unimodal stiffnesses of the components of the structures. However, the expressions are intractable in their exact form due to their inordinate length. It all has to do with the number of statically determinate substructures embedded in common engineering structures. This paper describes some preliminary results obtained from approximate analysis models for the internal forces using truncated expressions that are similar in form to the exact analytic ones. The approach is illustrated with numerical examples.

Introduction

THE classical methods of structural analysis methods are numerical techniques that compute the structural response to external loads for a given configuration of the structure and for given stiffnesses of its elements. Very often, engineers are faced with the need to reanalyze structures for the same geometrical configuration and for the same applied loads, but with modified stiffnesses. Such instances occur in structural design and, in particular, in optimal structural design where one usually tests many candidate stiffness distributions before an acceptable structure is found.[1] Structural reanalysis is also present in the analysis of framed structures when the P-δ effect is accounted for.[2] Since the axial load in a member modifies its flexural rigidity, such structures are solved iteratively. At a given iteration, the flexural stiffnesses of the members are computed, based on the axial loads of the previous iteration. The procedure is repeated until convergence.

When the stiffnesses of the individual members change, the analysis procedure must in principle be repeated with the modified parameters. There is no formula for the structural response in terms of the rigidities of its members. To alleviate the computational cost of multiple reanalysis, engineers have over the years developed reanalysis techniques that use results of previous analysis to predict the structural response when some or all of the rigidities are modified.[3,4] Noteworthy is the reciprocal approximation,[1] which expresses the nodal displacements as a linear Taylor polynomial in terms of the reciprocals of the member stiffnesses.

Recently, the first author has written a series of papers[5-7] describing the analytic solution of the structural analysis problem. For structures composed of uniform prismatic members of given geometry and assuming linear elastic behavior, the analytic expressions are exact equations for the inverse of the stiffness matrix, for the nodal displacements, and for the internal forces in the members, explicit in terms of the cross-sectional structural characteristics of the components of the structure. However, what seems to be Eldorado for the structural designer—an exact formula for the structural response—turns out to be practical only for relatively modest structures. In common engineering practice, the expressions, in their exact form, are very cumbersome. This is due to the fact that the expressions are ratios of polynomials, the number of terms of which is closely related to the number of determinate substructures that can be derived from the original one. This number is simply enormous.

However, for the first time, structural designers can contemplate the exact expressions for which they have been endeavoring to find approximate ones. Presumably, even with the exact analytic expressions at hand, structural analysts will still fall back on approximations for their reanalysis. But the analytic expressions may constitute the basis for a new generation of approximate models. Rather than using mathematical approximation techniques, one may very well conceive approximate models that emulate the construct of the analytic solution but retain only a fraction of the terms. The task is not trivial, for it assumes that a tractable number of terms in the series can stand in for the multitude of terms in the exact solution. This paper will delineate preliminary results obtained from analysis models that are based on such approximations. The models are tested on structural design problems that were previously treated in the literature. This does not imply that what is presented is convenient for automated structural design. At present, the purpose of the paper is to indicate a possible new path for developing approximate analysis models.

The theory was developed for linear elastic trusses. In the following section, we will show how structures composed of more complex elements can be cast in a truss-type formulation. Consequently, what will be presented herein is in principle valid for any finite element model in the linear elastic domain.

Structures Viewed as Generalized Trusses

A linear elastic truss of given geometry consisting of M members and N nodal degrees of freedom ($M \geq N$), which is subjected to an N vector of static loads $\boldsymbol{p}$ applied at the nodes of the structure, is governed by the following field equations of structural theory:

Equilibrium:

$$Qt = p \tag{1a}$$

Elasticity:

$$Se = t \tag{1b}$$

Compatibility:

$$Ru = e \tag{1c}$$

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where Q is the $N \times M$ equilibrium matrix, t the M vector of element axial forces, S the $M \times M$ unassembled stiffness matrix of the elements of the truss in deformation coordinates ($S_{ij} = \delta_{ij}s_j$), δ_{ij} the Kronecker delta, s_j ($=E_jA_j/L_j$) the element axial stiffness, E_j, A_j, and L_j Young's modulus, the cross-sectional area, and the length of element j, respectively, e the M vector of element deformations (total elongations), R the $M \times N$ compatibility matrix, and u the N vector of nodal displacements. Virtual work provides us with the statics-kinematics duality (SKD) in the form $Q^T = R$. Equations (1) are traditionally solved either by the stiffness method or by the flexibility method.

Similar equations can be written for structures composed of different types of elements. What sets the truss apart from most other structures is that the unassembled stiffness matrix S of a truss is strictly diagonal. This results from the truss element having one deformation coordinate only (axial extension). This property is shared with the prismatic torsional element. The size of an element stiffness matrix in deformation coordinates is equal to the number of nodal degrees of freedom minus the number of available equilibrium equations. The plane flexural element, for instance, has a 2×2 element stiffness matrix in deformation coordinates (4 displacements—2 equilibrium equations). Similarly, the size of the four-node plane stress element stiffness matrix in deformation coordinates is five (8 displacements—3 equilibrium equations). Since these matrices are usually nondiagonal, the unassembled stiffness matrix S of a structure is, in general, block diagonal.

To cast the field equations of any structure into the truss formulation [Eqs. (1)], we need a diagonal unassembled stiffness matrix. This can be achieved by expressing the element force-deformation relations [Eq. (1b)] in modal deformation coordinates by means of a congruent transformation based on the element modal matrix Φ_q

$$\hat{S}_q = \Phi_q^T S_q \Phi_q \tag{2}$$

where q is the element subscript and $\hat{S}_q$ and S_q are the element stiffness matrix in modal and in deformation coordinates, respectively. The modal stiffness matrix is obviously diagonal. Applying this technique to the uniform beam element[8] establishes that the beam element is composed of a unimodal pure moment component of stiffness EI/L and a pure shear component of stiffness $3EI/L$ that are attached in parallel to the element nodes. Along similar lines the four-node plane stress element can be represented by five unimodal components mounted in parallel: two flexural, a shear, a stretching, and a uniform extension element.[9]

This paves the way to treat a general finite element as a combination of unimodal, truss-like components. For our purposes, the explicit analysis equations that will be presented in the following apply to any linear elastic model. We will also assume that the field equations (1) are now written in modal coordinates, omitting the ^ superscript for clarity's sake.

Analytic Solution

Consider a structure composed of M unimodal components and N nodal degrees of freedom ($M \geq N$). Recall that a truss element introduces one unimodal component, a beam element contributes two unimodal components, and a four-node plane stress element, for instance, is represented by five unimodal components. Since we have only N equilibrium equations [Eq. (1a)], the degree of static redundancy of the structure is $R = M - N$. Consequently, we will have to use the elasticity equations [Eq. (1b)] and the compatibility equations [Eq. (1c)] to analyze the structure. In Refs. 5–7, the analytic solution of linear structural analysis was derived explicitly in terms of the unimodal stiffnesses of the components of the structure. We will indicate here the method to construct the solution. The interested reader can consult the aforementioned references for the formal proofs.

The most appealing expression is the formula for the internal forces. As a preparatory step, consider all of the statically determinate stable substructures embedded in the original structure. This amounts to selecting all of the $N \times N$ nonsingular submatrices Q_m in the $N \times M$ equilibrium matrix Q, with subscript m denoting a typical substructure. The column numbers of Q_m in Q define the elements that participate in the substructure. Let t_m be the vector of internal forces in the substructure if the substructure is to sustain alone the external loads p. The forces in the missing (redundant) elements are included in t_m and are given the value zero. Having computed the determinate forces in the substructures, the expression of the internal forces in the original structure is

$$t = \frac{1}{|K|}\sum_m |K_m| t_m \tag{3}$$

where $|K|$ is the determinant of the stiffness matrix of the structure and $|K_m|$ is the determinant of the stiffness matrix of substructure m. The summation runs over all of the statically determinate stable substructures. The equation gives the internal forces in a structure as a weighted sum of the forces in the individual determinate substructures. The weighting factor of every substructure is the ratio of the determinant of the stiffness matrix of the substructure to the determinant of the stiffness matrix of the original structure.

We choose to write the stiffness matrices of the original structure and of a typical substructure in the product form

$$K = QSR \tag{4}$$

and

$$K_m = Q_m S_m R_m \tag{5}$$

where S_m is the diagonal elasticity matrix of substructure m, and $R_m = Q_m^T$. The determinant of the stiffness matrix of a substructure is obviously

$$|K_m| = |Q_m||S_m||R_m| \tag{6}$$

Employing a theorem by Binet and Cauchy[10] the determinant of the stiffness matrix of the original structure can be expressed as

$$|K| = \sum_m |Q_m||S_m||R_m| \tag{7}$$

where the summation index m runs over all of the statically determinate stable substructures that can be derived from the redundant structure. Note, S_m being diagonal, its determinant is equal to the product of its diagonal elements

$$|S_m| = \pi_m = s_i s_j, \ldots, s_p \qquad (N \text{ terms}) \tag{8}$$

where the $i, j, \ldots, p$ subscripts correspond to the components of substructure m. Introducing Eqs. (6–8) in Eq. (3) and using the notation

$$B_m = |Q_m||R_m| = |Q_m|^2 \tag{9}$$

we obtain the expression of the internal force in component j of the structure explicitly in terms of the unimodal stiffnesses

$$t_j = \frac{\sum_m B_m t_{jm} \pi_m}{\sum_m B_m \pi_m} \qquad j = 1, \ldots, M \tag{10}$$

The internal force t_{jm} is the jth entry of vector t_m, and the π_m are the products of the stiffnesses of the N elements composing substructure m. The number of possible combinations of

N columns out of the M columns of $\boldsymbol{Q}$ is

$$C_N^M = \frac{M!}{N!R!} \tag{11}$$

however, not all of them correspond to stable structures. Consequently, the summation index in Eq. (10) runs over a subset of C_N^M. This concludes the assembly of the solution for the internal forces.

Focusing our attention to the inverse of the stiffness matrix, we note that it can be formulated as

$$\boldsymbol{K}^{-1} = \frac{\text{adj}\,\boldsymbol{K}}{|\boldsymbol{K}|} \tag{12}$$

where adj $\boldsymbol{K}$ is the adjoint matrix of $\boldsymbol{K}$. As in Eq. (10), the determinant of the stiffness matrix is

$$|\boldsymbol{K}| = \sum_m B_m \pi_m \tag{13}$$

We now need an explicit expression for the adjoint of $\boldsymbol{K}$. This can be obtained via the $(N-1)$ compounds of the matrices $\boldsymbol{Q}$ and $\boldsymbol{S}$.

The $(N-1)$ compound of $\boldsymbol{Q}$, denoted by $\boldsymbol{Q}^{(N-1)}$, is an $N \times C$ matrix where

$$C = C_{N-1}^M = \frac{M!}{(N-1)!(R+1)!} \tag{14}$$

The entries of the compound matrix are all of the possible $(N-1)$ minors of $\boldsymbol{Q}$. The row (column) position of the minors in the compound matrix is governed by the position of the rows (columns) of the minor in the original matrix, in ascending order. Similarly, the $(N-1)$ compound of $\boldsymbol{S}$, denoted by $\boldsymbol{S}^{(N-1)}$, is the $C \times C$ matrix whose entries are all of the $(N-1)$ minors of $\boldsymbol{S}$ in ascending order. It can be seen that $\boldsymbol{S}^{(N-1)}$ is a diagonal matrix whose main diagonal is populated with products of $(N-1)$ stiffnesses.

We now define a matrix $\boldsymbol{A}$ that is obtained by row inversion of $\boldsymbol{Q}^{(N-1)}$ and by multiplying all of the even numbered rows by (-1). Let $\boldsymbol{a}_n$ be column n of $\boldsymbol{A}$. Using the theorem of product of compound matrices by Binet and Cauchy and based on the definition of the adjoint of a matrix, it was shown[7] that the adjoint of the stiffness matrix is

$$\text{adj}\,\boldsymbol{K} = \sum_n \boldsymbol{a}_n \boldsymbol{a}_n^T \pi_n \tag{15}$$

where the summation index n is carried over all of the column indices of the $(N-1)$ compound of $\boldsymbol{Q}$. The π_n are products of $(N-1)$ stiffnesses in the corresponding positions of the $(N-1)$ compound of $\boldsymbol{S}$. Defining the generic matrix

$$\boldsymbol{A}_n = \boldsymbol{a}_n \boldsymbol{a}_n^T \tag{16}$$

and introducing Eqs. (13), (15), and (16) in Eq. (12) yields the explicit expression for the inverse of the stiffness matrix

$$\boldsymbol{K}^{-1} = \frac{\sum_n \boldsymbol{A}_n \pi_n}{\sum_m B_m \pi_m} \tag{17}$$

The modal displacements follow suit. Multiplying both sides of Eq. (17) by the applied loads, p produces

$$\boldsymbol{u} = \frac{\sum_n \boldsymbol{c}_n \pi_n}{\sum_m B_m \pi_m} \tag{18}$$

with

$$\boldsymbol{c}_n = \boldsymbol{A}_n \boldsymbol{p} \tag{19}$$

By multiplying both sides of Eq. (18) with the product $\boldsymbol{SR}$, it was shown that one obtains the expression of the internal forces given earlier in this section [Eqs. (3) and (10)].

Approximate Models Using Subsets of Polynomials

It is clear that the applicability of the analytic expressions hinges on the number of terms of the multilinear polynomials. The number of terms in the expression of the internal forces, for instance, is equal to the number of statically determinate stable substructures that can be derived from the original structure. This number is of the order of all of the possible combinations of N elements out of a total of M given by Eq. (11). The ratio of factorials in Eqs. (11) and (14) look benign, however, they run quickly out of control. Now, assuming that only a fraction of the substructures are statically stable, it seems hopeless to even start looking them up.

However, the explicit expressions may be found useful when considering a relatively small number of terms of the polynomials. In the case of the internal forces, one would therefore still use Eq. (10), but now the subscript m will run over a reduced set of statically determinate stable structures. We will here indicate a method to compute appropriate B_m coefficients. We have seen that in the exact expression the B_m are the squares of the determinants of the equilibrium matrices of the substructures. They can also be viewed as representing something more. Indeed, a perusal of Eq. (10) suggests a following interpretation for its constituents. The internal forces t_{jm} were obtained by using the equilibrium equations [Eq. (1a)]. The products of unimodal stiffnesses originate from the elasticity equations [Eq. (1b)]. It is natural to conjecture that the B_m constants stand for compatibility of deformations (1c). This is the procedure that was indeed followed in Ref. 6 to compute the B_m coefficients. Equation (10) gives us thus a beautifully structured view of structural analysis (Fig. 1).

Using the full series and enforcing global compatibility of deformations, that is, compatibility for all values of the stiffnesses, leads to an overdetermined but consistent set of linear equations in the B_m. In the case of a truncated expression for the internal forces, we note that equilibrium is still satisfied at all times. We can now compute the coefficients to ensure compatibility at preselected points in the design space, that is, pointwise compatibility. Since at these points all three field equations [Eqs. (1)] are satisfied, the truncated expression will yield exact results at the preselected points and will constitute an approximate model elsewhere. For every selected design point we can write R compatibility conditions. Because one of the B_m can be given an arbitrary value, by imposing exact results at P points we need $PR + 1$ terms in the expansion. The B_m coefficients will be determined by PR linear equations supplemented by, say, $B_1 = 1$.

The compatibility conditions appear as R homogeneous linear equations in the M modal deformations of the elements of the structure

$$\sum_{j=1}^{M} G_{ij} e_j = 0 \qquad i = 1, \ldots, R \tag{20}$$

where the G_{ij} are constants that can be computed from matrix $\boldsymbol{R}$. The expression for the element deformations is obtained by dividing both sides of Eq. (10) by the stiffness s_j of the corre-

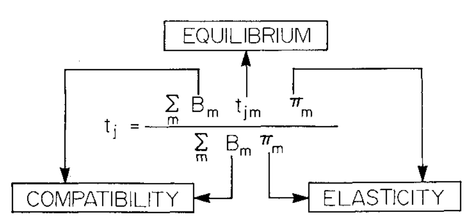

Fig. 1 Structured view of member forces.

sponding element for all of the components of the structure

$$e_j = \left(\sum_{m=1}^{PR+1} B_m t_{jm} \pi_m / s_j\right) \Bigg/ \left(\sum_{m=1}^{PR+1} B_m \pi_m\right) \quad j = 1, \ldots, M \tag{21}$$

Introducing Eqs. (21) in Eqs. (20), and multiplying throughout by the denominator of Eqs. (21), which is common to all of the terms, yields after grouping the expressions by B_m, R homogeneous equations, linear in the $PR + 1$ coefficients B

$$\sum_{m=1}^{PR+1} D_{im} B_m = 0 \qquad i = 1, \ldots, R \tag{22}$$

with

$$D_{im} = \sum_{j=1}^{M} G_{ij} t_{jm} \pi_m / s_j \tag{23}$$

We note that the coefficients in Eq. (22) depend on the unimodal stiffnesses of the structure, $D_{im}(s)$, via the products π_{mj}/s_j in Eq. (23). If we evaluate these coefficients at an arbitrary design point s_p, we obtain R conditions in order for the model of Eq. (10) to be exact at that point. By writing the compatibility conditions for P distinct points in the design space, we obtain, with $B_1 = 1$, $PR + 1$ linear equations in the B_m constants

$$\sum_{m=1}^{PR+1} D_{im}(s_p) B_m = 0 \qquad i = 1, \ldots, R; \quad p = 1, \ldots, P \tag{24}$$

$$B_1 = 1$$

Properties of the Approximate Models

The approximate models presented in this paper write the internal (modal) forces of a structure as a ratio of two multilinear polynomials in the unimodal stiffnesses of the elements of the structure [Eq. (10)]. Every term of the polynomials corresponds to a statically determinate substructure. The coefficients of the polynomials are computed by solving the set of linear equations (24). If all of the substructures are included in the polynomial expansions, the model is exact. Since this leads usually to more terms than can conceivably be handled, only a subset of the determinate substructures is considered, and as a result the model is approximate. At this stage, it is instructive to emphasize the following observations:

1) *The accuracy can be tuned.* The number of terms of the polynominals is arbitrary and can be varied at will. For every additional term in the expansion, one must supply a compatibility equation. Usually, however, one will increment the number of terms by multiples of the redundancy R since this allows one to render the model exact at specific points in the design space. Also, by writing compatibility conditions for points located in preferred regions, the model can be tailored to be more accurate in directions along which the design is most likely to proceed.

2) *The model is exact at specific points.* By writing R compatibility equations at preselected points, the model is exact at these points. The model gives also exact results for the determinate substructures included in the polynomials. This results from the fact that at least one redundant component of substructure k appears in the products π_m of all of the other substructures. Consequently, Eq. (10) becomes for substructure k

$$t_j(s_k) = \frac{B_k t_{jk} \pi_k}{B_k \pi_k} = t_{jk} \qquad j = 1, \ldots, M \tag{25}$$

For the same reason, the model will generate wrong results for structures close to a substructure that is not included in the polynomials. This is because a product π_m tends to zero as the design approaches a statically determinate solution different from substructure m.

3) *Scaling invariance.* The internal forces computed by the model are unaltered when the unimodal stiffnesses are multiplied by a positive constant α

$$t_j(\alpha s) = \frac{\sum_{m=1}^{PR+1} B_k t_{jm} \pi_m(\alpha s_m)}{\sum_{m=1}^{PR+1} B_m \pi_m(\alpha s_m)} = \frac{\alpha^N}{\alpha^N} t_j(s) = t_j(s) \quad j = 1, \ldots, M \tag{26}$$

That is, the forces generated by the model are homogeneous of degree 0 in the stiffnesses, a basic property of all structures. Note that this does not imply that scaling invariance exists in the design space.

4) *Interpolation.* For all structures, except those listed in observation 2, the model is approximate. It acts as an interpolation function, not unlike the interpolation functions used in finite element techniques. The construct of the interpolation is the same as in the exact model.

5) *Selection of the substructures.* In view of the immense number of available substructures, the selection of a subset that is to be retained for the model is usually a random process. However, some care must be exercised. It is a matter of good practice to distribute the occurrences of the elements as evenly as possible throughout the substructures. Also, an element must appear at least in one substructure, lest the solution will apply to a structure from which this element is missing. For reasons mentioned in observation 2, if the design process closes in on a statically determinate solution, the corresponding substructure should be part of the model. And last but not least, one must avoid using two substructures, say q and r, with identical internal forces ($t_{jq} = t_{jr}$, for all j). It is easy to verify that this leads to either a singular coefficients matrix or to a trivial solution of equations (24).

To test the validity of the model, one has to indulge in extensive numerical experimentation. We will report in the subsequent section some results obtained for unidirectional approximations.

Numerical Results

The proposed approximate models were verified for three planar structures: a 10-bar truss, a 20-bar semicircular cantilever truss, and a portal frame. Selecting two design points for each structure, s_0 and s_1, the approximate model was compared to the exact results for design points s on the line joining the two points

$$s = (1 - \lambda)s_0 + \lambda s_1 \tag{27}$$

where λ ranges from 0 to 1. In the analysis space, spanned over the unimodal stiffnesses, the approximate model is scaling invariant. To filter out the scaling effect, the error ϵ in the internal forces at s was plotted against the angular distance α between the initial design s_0 and s

$$\alpha = \text{arc cos} \frac{s_0 \cdot s}{\|s_0\| \, \|s\|} \tag{28}$$

where the numerator is the scalar product of vectors s_0 and s, and the denominator is the product of their Euclidean norms. The error ϵ_j for a typical internal force t_j is given by

$$\epsilon_j = (t_j^a - t_j)/t_j \tag{29}$$

where t_j^a is the approximate value of the internal force.

We have considered models based on exact analysis at one point (s_0), two points (s_0 and s_1), and three points, the two extreme points and an intermediate point half-way on the α scale. Using the compatibility equations (24), this lead to models with $(R + 1)$, $(2R + 1)$, and $(3R + 1)$ coefficients. The substructures were selected at random, although in some cases part of the substructures were imposed in order to check the contingency of the substructures on the quality of the approximations. Several numerical cases were run for the three test structures from which some common properties emerged. We will report herein the more significant results.

Ten-Bar Truss

This structure has been extensively treated in the literature under various loading conditions and constraints. We have selected the case of two equal downloads at the nodes of the bottom chord. The approximate models were tested on a line joining the initial design (20 in.^{2} for all cross sections) and the optimal design as reported in Ref. 1. The angular distance between the extreme points is about 45 deg. The 10-bar truss has $M = 10$ components, $N = 8$ degrees of freedom, and consequently, the structure has a static redundancy of $R = 2$. Using Eq. (11), we find that the truss has 45 determinate substructures, out of which 29 are stable.[5] The exact polynomials thus have 29 terms, and every term includes a product of eight stiffnesses, corresponding to the eight components of the specific substructure. The one-, two-, and three-point models lead to three, five, and seven linear equations, respectively, to compute the B_m coefficients.

The results are shown in Figs. 2–4. The one-point model in Fig. 2 gives large errors and was included for completeness sake only. Note that the model seems to be correct also in the vicinity of $\alpha = 17$ deg. Although similar behavior was detected in other cases, the reason is most probably fortuitous. The two-point model shown in Fig. 3 presents errors of up to 7%. The three-point model gives excellent results. As visualized in Fig. 4, the errors do not exceed 2% over the entire range. It should be noted that a 45-deg angular distance corresponds to a very substantial redesign. It is important to emphasize that the quality of the approximations is influenced by the set of substructures on which the model is based. A different set of base structures gave for the three-point case, for instance, less than 1% error.

Twenty-Bar Quarter-Circular Cantilever Truss

This structure is a modification of the circular arch treated in Ref. 11. Only the right part of the arch is considered, and consequently, it becomes a quarter-circular cantilever truss submitted to vertical and equal loads along the outer chord. This structure was selected to test a case with relatively many embedded substructures. The truss has $M = 20$ components, $N = 16$ degrees of freedom, and a degree of redundancy $R = 4$. The number of statically determinate subsets [Eq. (11)] is 4895, out of which 985 are stable. A two-point and three-point model will here be based on 9 and 15 coefficients, respectively, as compared to the 985 constants in the exact solution. A two-point model spanned over an angular distance close to 30 deg gave errors of up to 20%. A three-point model, shown in Fig. 5, yielded excellent results with errors of less than 0.5% over the entire range for all of the internal forces.

Three-Bar Portal Frame

The three-bar portal frame[12] is submitted to equal side loads as shown in Fig. 6. Every element is composed of three unimodal components: a moment, a shear, and an extension element, numbered in this order. This structure thus has $M = 9$ components, $N = 6$ degrees of freedom (two translations and one rotation at every node), and the degree of static redundancy is $R = 3$. From its 84 substructures, 46 were found to be stable. The angular distance of the extreme points is about 45 deg. The results for the three-point model (10 coefficients) are,

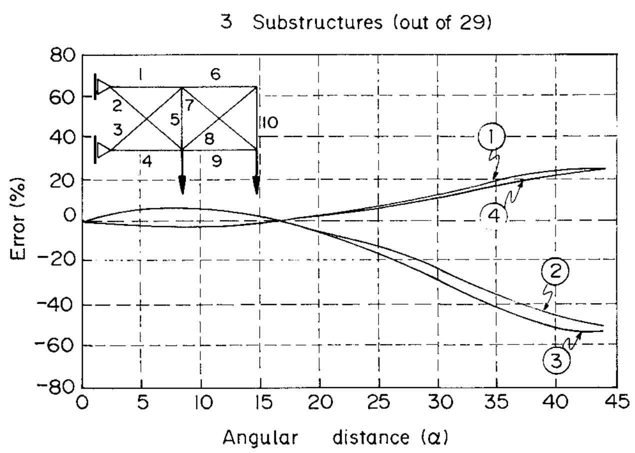

Fig. 2 Ten-bar truss: one-point model.

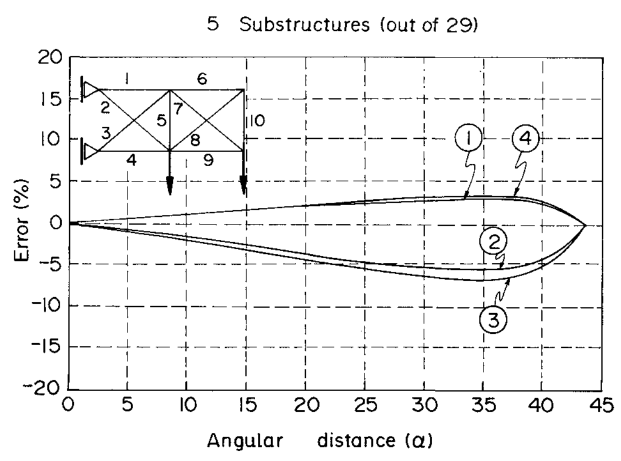

Fig. 3 Ten-bar truss: two-point model.

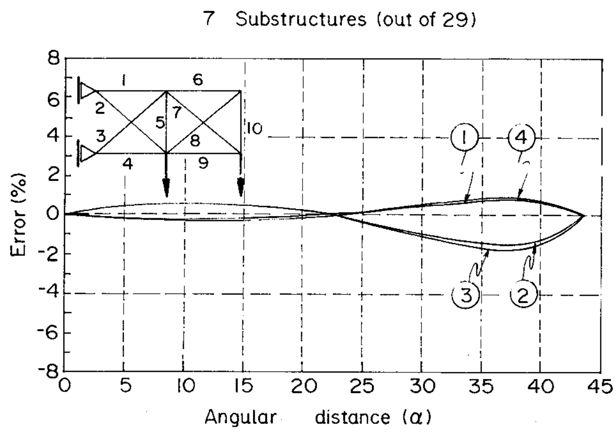

Fig. 4 Ten-bar truss: three-point model.

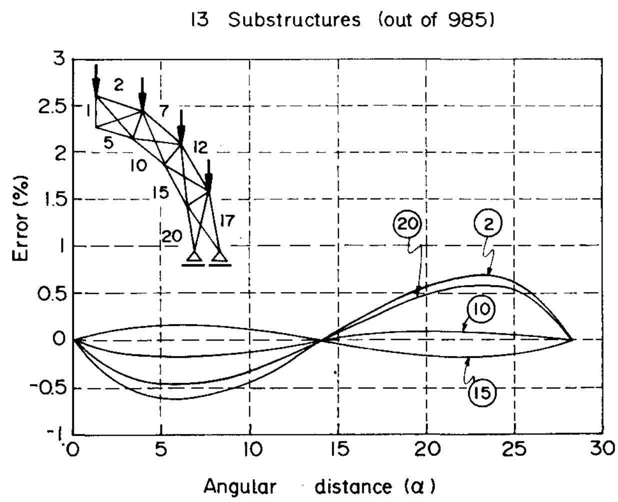

Fig. 5 Cantilever truss: three-point model.

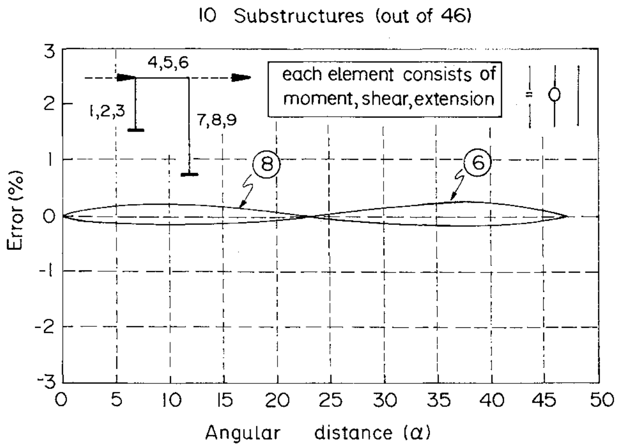

Fig. 6 Portal frame: three-point model.

for all practical reasons, perfect. Two-point models, based on seven coefficients, gave in one case a maximum of 15% error, although an error of 40% was found in another case.

Conclusions

This paper has described a first attempt to employ the exact analytic expression of the internal forces in structures composed of prismatic members as a basis for constructing explicit approximate analysis models. The technique consists in using the construct of the analytic expressions, ratios of multilinear polynomials in the unimodal stiffnesses, but to retain a very reduced set of terms in the polynomials. The method for determining the constants was to enforce compatibility of deformations at preselected points in the design space. Since equilibrium is always satisfied, the compatibility conditions make the model exact at the preselected points. Checking the approximations along a line joining two extreme designs, it was found that three-point models seem to yield excellent results. Since approximations of similar quality were obtained for the 10-bar truss (7 terms out of 29) and for the 20-bar cantilever truss (15 terms out of 985), results seem to indicate that the accuracy of the approximation is not expected to deteriorate with increasing number of elements.

These preliminary results are based on limited numerical testing. Before drawing significant conclusions, the scope of the applications should be broadened and more cases need to be investigated. In particular, one must consider models that predict the internal forces for random changes of the stiffnesses. Such aspects and others are the concerns of ongoing research aimed at generating high-quality approximate expressions for the internal forces employing ratios of multilinear polynomials in terms of the unimodal stiffnesses.

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